Advanced Trigonometry

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ADVANCED TRIGONOMETRY

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PREFACE

MOST teachers will agree that at the present time the work of mathematical specialists in schools is heavily handicapped by the absence of suitable text-books. There have been such radical changes in method and outlook that it has become necessary to treat large sections of some of the standard books merely as (moderately) convenient collections of examples and to supply the bookwork in the form of notes ; especially is this true of Algebra, Trigonometry, and the Calculus.

Dividing lines between these subjects tend nowadays to be obliterated. Methods of the Calculus are freely used in courses of Algebra and Trigonometry, while matter which used to find a place in the Algebra text-book is now included more conveniently elsewhere. Perhaps the most important example of this re-arrangement is the treatment of the logarithmic function. For many years past leading mathematicians have advocated a definition which transfers the chapter on the theory of logarithms from the Algebra to the Calculus text-book, and makes it the basis from which the exponential function is discussed, thus reversing the order commonly followed. The authors are convinced by their own experience that this is the best mode of approach. On general principles it would seem desirable also to follow the same order for the Complex variable, but unfortunately in practice this point of view appears to be too difficult for school work. By tradition the theory of the exponential and logarithmic functions of a complex variable is included in books on Advanced Trigonometry and this is a very reasonable arrangement ; it seems equally desirable to include also the theory of the corresponding functions of a real variable instead of relegating it to the Calculus book.

The interest and value of advanced trigonometry lies in regarding it as an introduction to modern analysis. The methods by which results are obtained are often more important—that is, educationally more valuable—than the results themselves. The character of the treatment in this book is shaped and controlled by that idea. Thus the methods for expanding functions in series focus attention on "remainders" and "limits" ; the methods for factorizing functions turn on establishing possible forms and then using the fundamental factor-theorem ; the discussion of complex numbers emphasises the fact that complex numbers are just as "real" as real numbers, etc. For the same reason no apology need be offered for the prevalence, in this book, of "inequalities." Their importance in higher mathematics can hardly be exaggerated, and they are invaluable too in elementary work. The "useful inequalities" of Chapter IV will, it is believed, be found fully worthy of their name.

The authors are planning text-books parallel to the present volume on Advanced Algebra and Calculus, written from a similar point of view. In all these subjects, it must be admitted, there are certain difficulties which the average student will never face, but which are all-important for the real mathematician; these include, for example, the purely arithmetical treatment of real number, limits, continuity, convergence, mean-value theorems, the analysis of area, length of a curve, etc. The authors propose to deal with these matters in a book which is cited as a "companion volume on Analysis," limiting the treatment, however, to what seems suitable for specialist work at schools. Although planned, no part of this book is yet written. The theory of Infinite Products has been left for this companion volume ; it is not so easy to provide a satisfactory *ab initio* treatment for products as it is for series and the alternative of taking for granted everything that really matters is undesirable. Happily also Infinite Products are of small value in elementary work and they are not required for most examinations. See. however, pp. 223, 240.

As a text-book on Trigonometry, this volume is a continuation of Durell and Wright's *Elementary Trigonometry*, and Chapter I should be regarded partly as a revision course. The sole object of Chapter XIV is to give opportunity for practice in mechanical manipulation to those who require it. The course really closes with Chapter XIII, which deals with a difficult subject and one which should be done carefully if it is done at all.

A Key is published, for the convenience of teachers, in which solutions are given in considerable detail, and in some cases alternative methods of solution are supplied, so that to some extent the *Key* forms a supplementary teaching manual.

The authors gratefully acknowledge help with the proofs received from Mr. J. C. Manisty, whose numerous criticisms and suggestions have enabled them to effect many improvements.

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CHAPTER I

PROPERTIES OF THE TRIANGLE

A LIST of the fundamental formulae connecting the elements of a triangle, proofs of which have been given in *Durell and Wright's Elementary Trigonometry*, will be found in Section D of the formulae at the beginning of that book ; references to these proofs will be indicated by the prefix *E.T.*

For geometrical proofs of theorems on the triangle, the reader is referred to some geometrical text-book. When these theorems are quoted or illustrated in this chapter, references, indicated by the prefix *M.G.*, are given to *Durell's Modern Geometry*.

Revision. Examples for the revision of ordinary methods of solving a triangle are given in Exercise I. a, below.

It is sometimes convenient to modify the process of solution. If, for example, the numerical values of b, c, A are given and if the value of a *only* is required, we may proceed as follows :

$$a^2 = b^2 + c^2 - 2bc\cos A;$$

$$\therefore\ a^2 = (b+c)^2 - 2bc(1+\cos A) = (b+c)^2 - 4bc\cos^2\tfrac{1}{2}A;$$

$$\therefore\ a^2 = (b+c)^2 - (b+c)^2\cos^2\theta,\ \text{ where }\ \cos^2\theta = \frac{4bc\cos^2\frac{1}{2}A}{(b+c)^2};$$

$$\therefore\ a = (b+c)\sin\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where

$$\cos\theta = \frac{2\sqrt{(bc)}\cos\frac{1}{2}A}{b+c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

θ is first found from (2) and then a is obtained from (1), both equations being adapted to logarithmic work.

An angle θ, used in this way, is called a subsidiary angle. For other examples of the use of subsidiary angles, see Ex. I. a, Nos. 21 to 25.

EXERCISE I. a.

[*Solution of Triangles*]

1. What are the comparative merits of the formulae for cos A, $\cos\frac{A}{2}$, $\sin\frac{A}{2}$, $\tan\frac{A}{2}$, when finding the angles of a triangle from given numerical values of a, b, c ?

2. Given $a = 100$, $b = 80$, $c = 50$, find A.

3. Given $a = 37$, $b = 61$, $c = 37$, find B.

4. Given $a = 11{\cdot}42$, $b = 13{\cdot}75$, $c = 18{\cdot}43$, find A, B, C.

5. Given A = 17° 55′, B = 32° 50′, $c = 251$, find a from the formula $a = c \sin A \operatorname{cosec} C$.

6. Given B = 86°, C = 17° 42′, $b = 23$, solve the triangle.

7. Given $b = 16{\cdot}9$, $c = 24{\cdot}3$, A = 154° 18′, find $\frac{1}{2}(B - C)$ from the formula $\tan \frac{1}{2}(B - C) = \frac{b - c}{b + c} \cot \frac{A}{2}$, and complete the solution of the triangle.

8. Given $b = 27$, $c = 36$, A = 62° 35′, find a.

Solve the triangles in Nos. 9-13 :

9. A = 39° 42′, B = 81° 12′, $c = 47{\cdot}6$.

10. $b = 6{\cdot}32$, $c = 8{\cdot}47$, B = 43°.

11. $a = 110$, $b = 183$, $c = 152$.

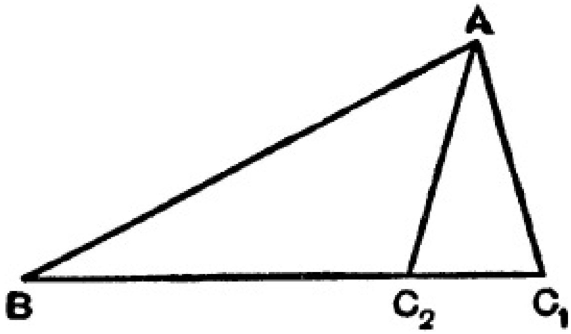
12. $a = 6{\cdot}81$, $c = 9{\cdot}06$, B = 119° 45′.

13. $b = 16{\cdot}9$, $c = 12{\cdot}3$, C = 51°.

[*The Ambiguous Case*]

14. Given A = 20° 36′, $c = 14{\cdot}5$, find the range of values of a such that the number of possible triangles is 0, 1, 2. Complete the solution if a equals (i) 8·3, (ii) 16·2, (iii) 3·2, (iv) 5·1.

15. Given b, c, and. B, write down the quadratic for a, and the sum and product of its roots, a_1 and a_2. Verify the results geometrically.



If A_1, C_1 and A_2, C_2 are the remaining angles of the two triangles which satisfy the data, find $C_1 + C_2$ and $A_1 + A_2$.

16. With the data of No. 15, prove that

$$\text{(i) } a_1 - a_2 = \pm 2\sqrt{(b^2 - c^2 \sin^2 B)}; \quad \text{(ii) } \sin \tfrac{1}{2}(A_1 - A_2) = \frac{a_1 - a_2}{2b}.$$

17. With the data of No. 15, prove that

$$(a_1 - a_2)^2 + (a_1 + a_2)^2 \tan^2 B = 4b^2.$$

18. (i) With the data of No. 15, if $a_1 = 3a_2$, prove that

$$2b = c\sqrt{(1 + 3 \sin^2 B)}.$$

(ii) With the data of No. 15, if $C_2 = 2C_1$, prove that

$$2c \sin B = b\sqrt{3}.$$

19. If the two triangles derived from given values of c, b, B have areas in the ratio 3 : 2, prove that $25(c^2 - b^2) = 24c^2 \cos^2 B$.

20. With the data of No. 15, if $A_1 = 2A_2$, prove that $4c^3 \sin^2 B = b^2(b + 3c)$.

$$4c^3 \sin^2 B = b^2(b + 3c).$$

[*Subsidiary Angles*]

21. Given $b = 16{\cdot}9$, $c = 24{\cdot}3$, $A = 154° 18'$, find a from formulae (1) and (2), p. 1.

22. Show that the formula $c = b \cos A \pm \sqrt{(a^2 - b^2 \sin^2 A)}$ may be written in the form $c = a \sin(\theta \pm A) \operatorname{cosec} A$, where $\sin \theta = \frac{b}{a} \sin A$.

23. Show how to apply the method of the subsidiary angle to $a^2 = (b - c)^2 + 2bc(1 - \cos A)$.

24. In any triangle, prove that $\tan \frac{1}{2}(B - C) = \tan(45° - \theta) \cot \frac{1}{2}A$, where $\tan \theta = \frac{c}{b}$.

Hence find $\frac{1}{2}(B - C)$ if $b = 321$, $c = 436$, $A = 119° 15'$.

25. Express $a \cos \theta - b \sin \theta$ in a form suitable for logarithmic work.

[*Miscellaneous Relations*]

26. If $a = 4$, $b = 5$, $c = 6$, prove that $C = 2A$.

27. Express in a symmetrical form $\frac{a}{bc} + \frac{\cos A}{a}$.

28. Prove that $b^2(\cot A + \cot B) = c^2(\cot A + \cot C)$.

29. Simplify $\operatorname{cosec}(A - B) \,.\, (a \cos B - b \cos A)$.

30. Prove that $a^2 \sin(B - C) = (b^2 - c^2) \sin A$.

31. Prove that $\frac{b \sec B + c \sec C}{\tan B + \tan C} = \frac{c \sec C + a \sec A}{\tan C + \tan A}$.

32. If $b \cos B = c \cos C$, prove that either $b = c$ or $A = 90°$.

33. Prove that $\sin^2 A + \sin B \sin C \cos A = \frac{2\Delta^2(a^2 + b^2 + c^2)}{a^2 b^2 c^2}$.

34. Prove that $\frac{1 + \cos(A - B) \cos C}{1 + \cos(A - C) \cos B} = \frac{a^2 + b^2}{a^2 + c^2}$.

35. Prove that

$$a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \frac{2\Delta \sin A}{a}.$$

36. Express $\cos \frac{1}{2}(A - B) \,.\, \operatorname{cosec} \frac{C}{2}$ in terms of a, b, c,

37. If $b + c = 2a$, prove that $4\Delta = 3a^2 \tan \frac{A}{2}$.

38. If $a^2 = b(b + c)$, prove that A = 2B.

39. Prove that $c^2 = a^2 \cos 2B + b^2 \cos 2A + 2ab \cos(A - B)$.

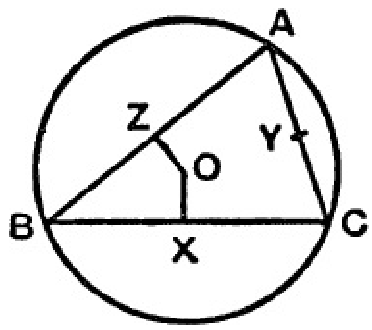
40. Prove that $\frac{b-c}{b+c} \cot \frac{A}{2} + \frac{b+c}{b-c} \tan \frac{A}{2} = 2 \operatorname{cosec} (B - C)$.

41. Prove that

$$a(1 + 2\cos 2A)\cos 3B + b(1 + 2\cos 2B)\cos 3A = c(1 + 2\cos 2C).$$

42. If cos A cos B + sin A sin B sin C = 1, prove that A = 45° = B.

The Circumcentre. The centre O of the circle through A, B, C is found by bisecting the sides of the triangle at right angles, and the radius is given by the formulae



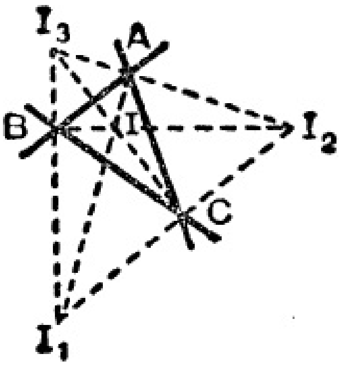
$$R = BX \operatorname{cosec} BOX = \frac{a}{2 \sin A}; \quad \text{.........(3)}$$

$$\therefore \ R = \frac{abc}{2bc \sin A} = \frac{abc}{4\Delta}. \quad \text{.....................(4)}$$

The reader should prove that these formulae hold also when ∠ BAC is obtuse.

The in-centre and e-centres. The centres I, I_1, I_2, I_3 of the circles which touch the sides are found by bisecting the angles of the triangle, internally and externally.

FIG. 2.



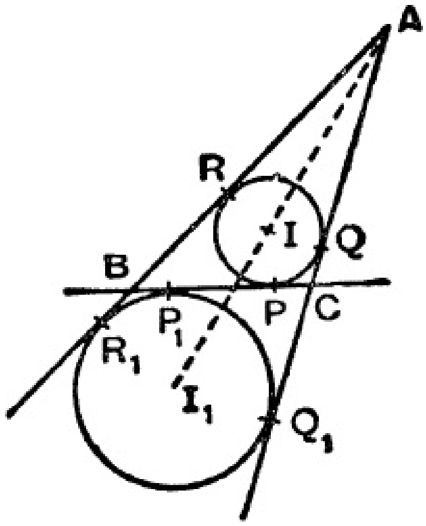


FIG. 3.

The radii of these circles are given by

$$r = \frac{\Delta}{s}; \quad r_1 = \frac{\Delta}{s-a}, \text{ etc.} \quad \ldots\ldots\ldots\ldots(5)$$

$$r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}; \quad r_1 = 4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}, \text{ etc.} \ldots\ldots(6)$$

Also in Fig. 3, we have

$$AR = s - a; \quad AR_1 = s; \quad BP_1 = s - c; \quad \ldots\ldots\ldots\ldots(7)$$

$$\therefore \ r = (s-a)\tan\frac{A}{2}; \quad r_1 = s\tan\frac{A}{2}. \quad \ldots\ldots\ldots\ldots(8)$$

For proofs of these formulae and further details, see *E.T.*, pp. 184·186, 277, 278 and *M.G.*, pp. 11, 24, 25.

The Orthocentre and Pedal Triangle. The perpendiculars AD, BE, CF from the vertices of a triangle to the opposite sides meet at a point H, called the orthocentre ;

the triangle DEF is called the pedal triangle (*M.G.*, p. 20).

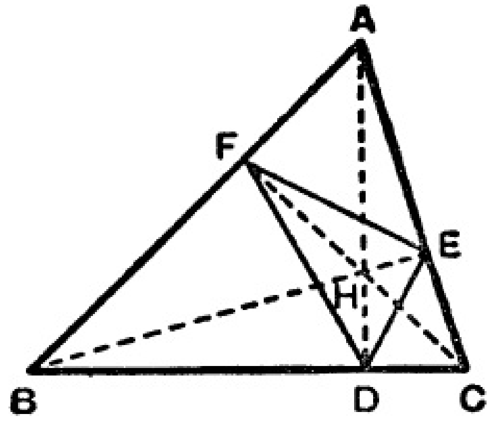


FIG. 4.

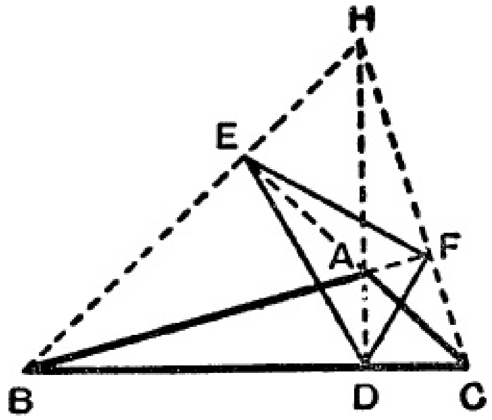


FIG. 5.

If Δ ABC is *acute-angled*, (Fig. 4), H lies inside the triangle.
Since BFEC is a cyclic quadrilateral, AFE and ACB are similar triangles ;

$$\therefore \frac{EF}{BC} = \frac{AF}{AC} = \cos A;$$

$$\therefore EF = a \cos A. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Since HECD is a cyclic quadrilateral, ∠ HDE = ∠ HCE = 90° – A ; similarly ∠ HDF = 90° – A ;

$$\therefore \angle EDF = 180° - 2A. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Further, HD bisects ∠ EDF and similarly HE bisects ∠ DEF ; ∴ H is the in-centre of Δ DEF. Also since BC is perpendicular to AD, it is the external bisector of ∠ EDF ; hence A, B, C are the e-centres of the pedal triangle.
We have also

$$AH = AE \operatorname{cosec} AHE = c \cos A \operatorname{cosec} C = 2R \cos A, \quad \ldots\ldots\ldots (11)$$

and

$$DH = BH \cos BHD = 2R \cos B \cos C. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

The reader should work out the corresponding results for Fig. 5, where the triangle is *obtuse-angled*.

If ∠ BAC is obtuse, ∠ EDF = 2A – 180° and other results are modified by writing – cos A for cos A. [See Ex. I. b, No. 27 and note the difference of form in No. 36. See also *Example* 3.]

The Nine-Point Circle. The circle which passes through the midpoints X, Y, Z of the sides BC, CA, AB passes also through D, E, F and through the mid-points of HA, HB, HC ; it is therefore called the nine-point circle and its centre N is the mid-point of OH (*M.G.*, p. 27).

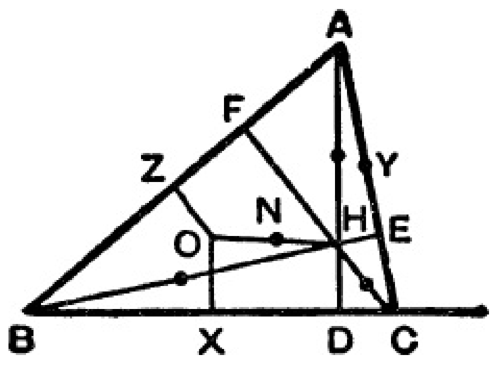


FIG. 6.

Since Δ XYZ is similar to Δ ABC and of half its linear dimensions, the radius of the nine-point circle is $\frac{1}{2}$R.

Since each of the points H, A, B, C is the orthocentre of the triangle formed by the other three, the circumcircle of Δ DEF is the common nine-point circle of the four triangles ABC, BCH, CHA, HAB.

Also, since Δ ABC is the pedal Δ of Δ $I_1I_2I_3$ and of Δ II_2I_3, etc., the circumradius of each of these triangles is 2R.

The Polar Circle. In Fig. 6 and Fig. 7 we have, by cyclic quadrilaterals,

$$HA \,.\, HD = HB \,.\, HE = HC \,.\, HF.$$

In Fig. 7, where ∠ BAC is *obtuse*, A and D are on the same side of H, and so also are B, E and C, F. In this case, if HA . HD = p^2, it follows that the polars of A, B, C w.r.t. the circle, centre H, radius ρ, are BC, CA, AB.

The triangle ABC is therefore self polar w.r.t. this circle ; and the circle is called the polar circle of Δ ABC.

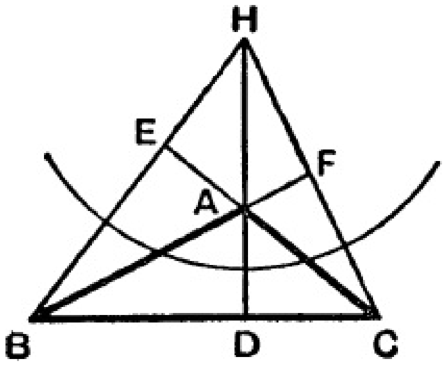


FIG. 7.

We have

$$\rho^2 = HA \, . \, HD = (-2R\cos A)(2R\cos B\cos C);$$

$$\therefore \quad \rho^2 = -4R^2 \cos A \cos B \cos C. \; \ldots (13)$$

An acute-angled triangle in real geometry has no polar circle.

Notation. The lettering already adopted for special points connected with the triangle will be employed throughout the Chapter. This will shorten the statement of many of the examples.

We add some illustrative examples.

Example 1. Prove $s^2 = \Delta(\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C)$.

Since

$$\tfrac{1}{2}(A + B + C) = 90°,$$

$$\Sigma \cot \tfrac{1}{2}A = \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C, \text{ (see } E.T.\text{, p. 272, Ex. V.)}$$

$$\therefore \quad \Sigma \cot \tfrac{1}{2}A = \sqrt{\left\{\frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-c)(s-a)} \cdot \frac{s(s-c)}{(s-a)(s-b)}\right\}} = \frac{s^2}{\Delta}.$$

Example 2. Express $\dfrac{(ab - r_1 r_2)}{r_3}$ in a symmetrical form.

Since

$$r_1 r_2 = \frac{\Delta^2}{(s-a)(s-b)} = s(s-c),$$

$$\begin{aligned} 4ab - 4r_1 r_2 &= 4ab - (a+b+c)(a+b-c) \\ &= c^2 - (a-b)^2 \\ &= 4(s-a)(s-b); \end{aligned}$$

$$\therefore \frac{(ab - r_1 r_2)}{r_3} = \frac{(s-a)(s-b)}{r_3}$$

$$= \frac{(s-a)(s-b)(s-c)}{\Delta} = \frac{\Delta}{s}.$$

Example 3. If J is the in-centre of BHC, express the radius of the circle BJC in terms of R and A.

By equation (3) the radius is $\frac{1}{2}$BC cosec BJC, but

$\angle$ BJC = 90° + $\frac{1}{2}\angle$ BHC = 180° – $\frac{1}{2}$A, if B and C are acute angles;

$$\therefore \text{ the radius} = \frac{a}{2 \sin \frac{1}{2}A} = \frac{2R \sin A}{2 \sin \frac{1}{2}A} = 2R \cos \tfrac{1}{2}A.$$

If either B or C is obtuse, $\angle$BJC = 90° + $\frac{1}{2}$A, and then the radius = 2R sin $\frac{1}{2}$A.

EXERCISE I. b.

1. If a = 15·1, A = 24° 36′, find R.
2. If a = 3, b = 5, c = 7, find R and r.
3. If a = 13, b = 14, c = 15, find r_1, r_2, r_3.
4. If a = 23×5, A = 62°, and b = c, find R and r.
5. Prove that

$$\text{(i) } \angle BAI_3 = 90° - \frac{A}{2} = \angle I_3 I_1 I_2 ; \quad \text{(ii) } II_1 = 4R \sin \frac{A}{2};$$

$$\text{(iii) } I_2 I_3 = a \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{A}{2}.$$

6. Verify Equation (6), p. 4, by using the formulae $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, etc., in terms of the sides.

7. Express a (cos A + cos B cos C) in a symmetrical form.

Prove the following relations :

8. $s = 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

9. $s - a = 4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

10. $r_2 r_3 \tan \frac{1}{2} = \Delta$.

11. $r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2$.

12. $r^2 + r^3 = 4R \cos^2 \frac{A}{2}$.

13. $r - r_1 + r_2 + r_3 = 2a \cot A$.

14. $AI \,.\, AI_1 = bc$.

15. IA. IB = 4Rr $\sin \frac{C}{2}$.

16. IA. IB. IC = $\frac{abc\Delta}{s^2}$.

17. $II_1 . II_2 . II_3 = 16R^2 r$.

18. Δ ABI : Δ ACI = $c : b$.

19. $AD^2 (\cot B + \cot C) = 2\Delta$.

20. AD = $2r \operatorname{cosec} \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

21. $\Delta OI_2I_3 : \Delta OI_3I_1 = (b + c) : (a + c)$.

22. AH = α cot A = 2OX.

23. AH + BH +CH = 2 (R + r).

24. If $a = 14$, $b = 13$, $c = 15$, prove that AD = 12.

25. Given B = 37°, C =46°, BE = 9×3, find 6.

26. If BP . PC = Δ, (see Fig. 3), prove that A =90°.

27. In Fig. 5, where ∠BAC is obtuse, prove that

(i) EF = $-a \cos A$, FD = $b \cos B$, DE = $c \cos C$;

(ii) ∠FDE = 2A – 180°, ∠ DEF = 2B, ∠ EFD = 2C;

(iii) AH = – 2R cos A, BH = 2R cos B, CH = 2R cos C;

(iv) HD = 2R cosB cos C, HE = – 2R cos C cos A,
HF = – 2R cos A cos B.

28. If $a = 13$, $b = 9$, $c = 5$, find ρ (see p. 6).

29. Find an expression for the radius of the polar circle of $\Delta II_2 I_3$ in terms of R, r_1.

30. Prove that the circumradius of Δ HBC equals R.

31. Prove that the circumradius of Δ OBC is > $\frac{1}{2}$R.

32. Prove that the in-radius of Δ AEF is r cos A.

33. Prove that the area of Δ DEF is ± 2A cos A cos B cos C.

34. Given b, c, B, prove that the product of the in-radii of the two possible triangles is $c(c - b) \sin^2 \frac{1}{2}B$.

35. Prove that the in-radius of $\Delta I_1I_2I_3$, is $2R\left\{\Sigma\left(\sin\frac{A}{2}\right) - 1\right\}$.

36. If Δ ABC is acute-angled, prove that the perimeter of Δ DEF is 4R sin A sin B sin C. If ∠ BAC is obtuse, prove that the perimeter is 4R sin Δ cos B cos C.

37. Find in terms of A, B, C, R the in-radius of Δ DEF (i) if Δ ABC is acute-angled, (ii) if ∠ BAC is obtuse.

38. Prove that $a \sin B \sin C + b \sin C \sin A + c \sin A \sin B = \frac{3\Delta}{R}$.

39. Express $\frac{\Delta}{a} + r \cos A - R \cos^2 A$ in a symmetrical form.

40. Prove that

(i) $a^2 \cos^2 A = b^2 \cos^2 B + c^2 \cos^2 C + 2bc \cos B \cos C \cos 2A$;

(ii) $a^2 \cos^2 A \cos^2 2A = b^2 \cos^2 B \cos^2 2B + c^2 \cos^2 C \cos^2 2C + 2bc \cos B \cos C \cos 2B \cos 2C \cos 4A$;

(iii) $a^2 \operatorname{cosec}^2 \frac{A}{2} = b^2 \operatorname{cosec}^2 \frac{B}{2} + c^2 \operatorname{cosec}^2 \frac{C}{2} - 2bc \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2} \sin \frac{A}{2}$.

Any Line through a Vertex. Suppose any line through A cuts BC at K. Denote $\frac{BK}{KC}$ by $\frac{z}{y}$, so that K is the centroid of masses y, z at B, C respectively.

Let ∠ BAK = β, ∠ KAC = γ, ∠AKC = θ.

Draw BB′, CC′ perpendicular to AK.

Then

$$\frac{BK}{KC} = \frac{BB'}{CC'} = \frac{c \sin \beta}{b \sin \gamma}. \qquad \text{............(14)}$$

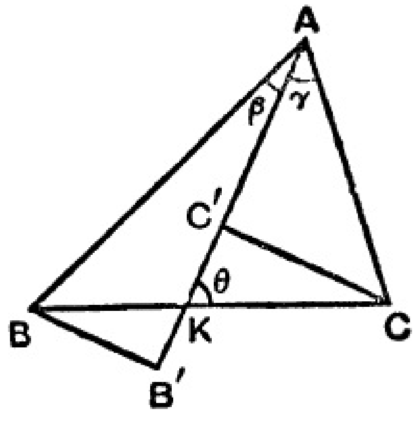


FIG. 8.

This may be written